J. Fluid Mech. (1968), *vol.* 34, *part* 3, *pp.* 609–624
Printed in Great Britain

Reflexion and stability of waves in stably stratified fluids with shear flow: a numerical study

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(Received 5 February 1968)

A numerical examination has been made of the reflectivity of critical levels with low Richardson number to internal gravity waves propagating in stratified fluids with shear. At sufficiently low positive Richardson numbers the reflected wave may actually be stronger than the incident.

The normal mode instabilities of three simple models have also been computed. The results are presented in three dimensions: Richardson number, horizontal wave scale and real wave frequency.

1. Introduction

This study was originally undertaken as a numerical extension of the works of Bretherton (1966) and Booker & Bretherton (1967) on the propagation of internal gravity waves through critical levels in stably stratified shearing fluids. A critical level is a level at which the horizontal trace velocity of the wave equals the mean horizontal velocity of the fluid. These authors found that at a Richardson number R, large compared to $\frac{1}{4}$, very little wave momentum and energy are transmitted, and also very little are reflected. Bretherton discussed the results in terms of a wave packet continually moving closer to, but never reaching, the critical level.

These studies, particularly with regard to the reflexion of waves, have here been extended to the range of Richardson numbers 0–$\frac{1}{4}$. In this range wave behaviour can be quite different from that for larger values of R.

In these numerical analyses, a periodic wave was considered to be generated at some remote depth, and to propagate upward through a region of constant shear until it reached a critical level. At some point above the critical level, the shear zone was terminated and the model capped with an infinite atmosphere of uniform mean velocity.

It was found that a substantial amount of reflexion occurs. At any given wave-number and frequency, measured relative to the top fluid, there is a critical value of R below which the reflected wave is actually larger than the incident wave. This condition, which will be called *over-reflexion*, stems from the ability of the wave to extract energy and momentum from the mean flow. The critical value of R for total reflexion depends, among other factors, on whether the transmitted wave is evanescent or propagating in the upper fluid region. Values ranged up to $\frac{1}{4}$ in the former case, but did not exceed 0·115 in the latter.

If a wave is over-reflected, it is possible that the wave source, for example a layer of turbulence, can in the net receive rather than give up energy. A knowledge of the reflectivity of the mean fluid wind structure may thus provide insight about the larger scales at which energy is fed into turbulence as well as to the energy losses to propagating internal gravity waves. These are the scales of the *buoyancy subrange* of turbulence discussed by Bolgiano (1962) and Lumley (1964).

In a sense, then, the reflectivity results are pertinent to non-linear instability, i.e. turbulence in the fluid. At low Richardson numbers the fluid may also be linearly unstable with exponentially growing normal modes. These will occur if the wind structure and boundary conditions are suitable. The mathematical analyses for the wave propagation and stability problems are very similar, the former involving the response to an inhomogeneous forcing, while the latter are the homogeneous solutions to the same equations. It was thus a relatively straightforward task to extend these computations and to examine the stability of two or three types of atmospheric models.

One result of the stability computations is to inject a note of caution in the use of singular neutral modes in defining domains of instability. Miles (1961) has already cited one such caution: instability domains are bounded by singular neutral modes, but the converse need not hold. The point to be made here is that these instability domains exist in a multi-dimensional space and may need to be considered in such a context.

As Miles has shown, the problem can be described in terms of several parameters, including wave-number, complex wave speed (or wave frequency, as we shall use), Richardson number, and some measure of scale, such as the ratio of shear zone thickness to scale of density variation. Unstable eigensolutions involving a complex part of the wave speed may be represented as hypersurfaces in the four-dimensional space with Richardson number, real wave speed (or real frequency), wave-number, and scale measure as axes.

These hypersurfaces are indeed bounded by singular neutral modes. However, what has often been considered is the two-dimensional projection of the singular neutral modes onto the Richardson number–wave-number plane. What is not *a priori* evident is that the *projection* of the instability domains onto such a plane is bounded by the *projection* of the singular neutral modes.

Three models have been analysed: (i) a zone of constant shear bounded below by a solid surface and above by an infinite depth of fluid with constant velocity; (ii) a similar model but with a region of constant velocity between the ground and the shear layer; (iii) a shear layer bounded above and below by infinitely thick layers of constant velocity. In all cases the fluid was taken to be incompressible, with an exponential variation in mean density throughout. The difference in velocity between top and bottom (our scale measure) was held constant, while Richardson number, wave-number, and real frequency were varied.

The first model showed no instability, while the latter two had instability surfaces of a reasonably complex nature. The second was unstable at all wave-numbers at sufficiently low Richardson number, and the third was unstable only for large wave-numbers.

2. Reflexion

The mathematical model

We shall consider a fluid that is incompressible, inviscid and adiabatic, and has a mean density ρ_0 which varies as $e^{-\beta z}$, z being the vertical co-ordinate. The geometry is taken as Cartesian and non-rotating, with a downward gravitational acceleration g. Mean values of variables are denoted with a subscript 0, while wave perturbations are given without subscript.

We shall assume that above $z = 0$ the fluid has no mean velocity, and that below $z = 0$ and extending some unspecified distance down there is a velocity $u_0 = u_0' z$ in the x-direction. The shear, u_0', is taken as constant and positive in this region.

The wave perturbations are taken to vary only in the x- and z-directions. It can be shown that variations in the y-direction simply increase the effective value of the Richardson number. Then the perturbation equations of motion are:

$$\frac{\partial u}{\partial t}+u_0\frac{\partial u}{\partial x}+w\frac{\partial u_0}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{1}$$

$$\frac{\partial w}{\partial t}+u_0\frac{\partial w}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g\frac{\rho}{\rho_0}, \tag{2}$$

where w is the vertical velocity and p the pressure. Incompressibility requires that

$$\frac{\partial \rho}{\partial t}+u_0\frac{\partial \rho}{\partial x}+w\frac{\partial \rho_0}{\partial z} = 0, \tag{3}$$

and the conservation of mass that

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0. \tag{4}$$

Assuming the perturbation quantities vary as $\exp\{i(\omega t+kx)\}$, these equations can be combined to yield (Chandrasekhar 1961)

$$\frac{\partial^2 w}{\partial z^2}-\beta\frac{\partial w}{\partial z}+\left[\frac{g\beta k^2}{(\omega+ku_0)^2}-k^2+\frac{\beta k}{(\omega+ku_0)}\frac{du_0}{dz}-\frac{k}{(\omega+ku_0)}\frac{d^2u_0}{dz^2}\right]w = 0. \tag{5}$$

Note that the frequency, ω, refers to a frame of reference moving with the upper half space.

If we make the substitutions

$$\left.\begin{aligned} w &= w_1 e^{\frac{1}{2}\beta z}, & N^2 &= g\beta, & \zeta &= (4k^2+\beta^2)^{\frac{1}{2}}z, \\ R &= N^2/u_0'^2, & \Omega &= \omega/N, & j &= \beta/(4k^2+\beta^2)^{\frac{1}{2}}, \end{aligned}\right\} \tag{6}$$

then for $z > 0$, this becomes

$$\frac{\partial^2 w_1}{\partial \zeta^2}+\left[-\frac{1}{4}+\frac{1}{4\Omega^2}\left(1+\frac{\beta^2}{4k^2}\right)^{-1}\right]w_1 = 0, \tag{7}$$

and for $z < 0$
$$\frac{\partial^2 w_1}{\partial \zeta^2} + \left[-\frac{1}{4} + \frac{j}{(\zeta+\zeta_0)} + \frac{R}{(\zeta+\zeta_0)^2} \right] w_1 = 0, \tag{8}$$

where
$$\zeta_0 = \frac{2\omega}{u_0'}\left(1+\frac{\beta^2}{4k^2}\right)^{\frac{1}{2}}. \tag{9}$$

In the upper half space, solutions take the form

$$w_1 = C\,e^{i\lambda\zeta} + D\,e^{-i\lambda\zeta}, \tag{10}$$

where C and D are integration constants, and

$$\lambda = \frac{1}{2}\left[-1 + \frac{1}{\Omega^2}\left(1+\frac{\beta^2}{4k^2}\right)^{-1}\right]^{\frac{1}{2}}. \tag{11}$$

In the lower half space, equation (8) is Whitaker's equation and the solution may be written in terms of Whitaker's functions (cf. Slater 1960):

$$w_1 = AM_{j,m}(\zeta+\zeta_0) + BM_{j,-m}(\zeta+\zeta_0), \tag{12}$$

where A and B are integration constants and

$$m = (\tfrac{1}{4} - R)^{\frac{1}{2}}, \tag{13}$$

$$M_{a,b}(y) = y^{\frac{1}{2}+b}\,e^{-\frac{1}{2}y}\left[1 + \frac{\frac{1}{2}+a-b}{1\,!(2m+1)}y + \frac{(\frac{1}{2}+a-b)(\frac{3}{2}+a-b)}{2\,!(2m+1)(2m+2)}y^2 + \dots\right]. \tag{14}$$

Boundary conditions

Two boundary conditions need to be met at the interface at $\zeta = 0$. To preserve continuity of vertical displacement, w_1 must be continuous across the interface. The second condition can be obtained by integrating (5) from $z = -\epsilon$ to $z = +\epsilon$ and passing to the limit $\epsilon = 0$, yielding the continuity of

$$\omega\frac{\partial w_1}{\partial z} - ku_0' w_1. \tag{15}$$

This may be written
$$\left[\frac{\partial w_1}{\partial \zeta} - \frac{w_1}{\zeta_0}\right]_{\zeta=\zeta_{0-}} = \left[\frac{\partial w_1}{\partial \zeta}\right]_{\zeta=\zeta_{0+}}. \tag{16}$$

An additional upper boundary condition is needed as $\zeta \to \infty$. For this problem, we are concerned only with real values of frequency so that λ is either pure real or pure imaginary. If λ is imaginary, so that the transmitted wave is evanescent in the upper half space, only the mode which decays with increasing height is allowed. Thus if

$$\lambda = i\mu, \tag{17}$$

where μ is positive,
$$w_1 = C\,e^{-\mu\zeta} \tag{18}$$

in the upper half space. The physical implication is that the wave is totally reflected in the evanescent region.

If λ is real, the transmitted wave propagates in the top half space. A radiation boundary condition is then imposed; only the mode providing an upward wave energy flux is present. The wave energy flux is

$$\overline{p_1 w_1} = \tfrac{1}{2}\mathrm{Re}\,p_1 w_1^* = -\left(\frac{\omega}{k} + u_0\right)\left(1+\frac{\beta^2}{4k^2}\right)^{\frac{1}{2}} \mathrm{Re}\left(iw_1^* \frac{dw_1}{d\zeta}\right), \tag{19}$$

where the overbar denotes an average over a cycle and the asterisk denotes complex conjugate. The radiation boundary condition then requires that

$$w_1 = C\,e^{i\lambda\zeta} \tag{20}$$

in the top half space.

The upper and interface boundary conditions are sufficient to determine the ratio of the amplitude constants A and B of (12). This is sufficient to determine the sign of $\overline{p_1 w_1}$ throughout the fluid, as will be shown. The amplitude of $\overline{p_1 w_1}$ and w_1 would be fixed if an inhomogeneous boundary condition were to be provided as, for example, by specifying the amplitude of w_1 at some lower height.

The wave Reynolds stress, or vertical flux of horizontal momentum, is

$$\overline{u_1 w_1} = \left(1+\frac{\beta^2}{4k^2}\right)^{\frac{1}{2}} \mathrm{Re}\left(iw_1^* \frac{dw_1}{d\zeta}\right). \tag{21}$$

It has been shown that except at a singular level, $\overline{u_1 w_1}$ is independent of height (Eliassen & Palm 1961). In the shear zone, where equation (8) applies,

$$\left.\begin{aligned} \mathrm{Re}\left(iw_1^* \frac{dw_1}{dz}\right) &= -2m\mathrm{Re}(iA^*B), \quad (\zeta+\zeta_0) > 0, \\ &= -2m\mathrm{Re}(iA^*B)\,e^{i2\pi m}, \quad (\zeta+\zeta_0) < 0. \end{aligned}\right\} \tag{22}$$

This discontinuity has been noted by Booker & Bretherton (1967) in connexion with momentum flux discontinuity at the singular level. In order to obtain (22) it is necessary to choose a branch of the solution at $\zeta+\zeta_0 = 0$. We follow Booker & Bretherton in choosing the branch for which $\arg(\zeta+\zeta_0) = -\pi$, $\zeta+\zeta_0 < 0$, referring the reader to their arguments.

Since $(\omega+ku_0)$ changes sign at the singular level, $\overline{pw}$ has the same sign throughout the fluid on either side of the singular level, though it may have a different sign on one side than the other.

If $\overline{pw}$ is positive below the singular level, then a wave source located farther down sees the level as at most partially reflecting. If $\overline{pw} = 0$ the level is seen as totally reflecting, and if $\overline{pw} < 0$, the level is seen as over-reflecting.

Numerical computations

By inspection of (7), (8), (9), (16), (18) and (20), one sees that there are three parameters on which reflectivity must depend: R (or m), j and Ω. Note that Ω is a measure of the frequency of the wave in relation to the fluid in the upper half space. Also, j is a measure of the horizontal scale of the wave; it approaches zero for very short wavelengths and one for very long wavelengths. Reflectivity also depends on ζ_0, but this may be written in terms of the other parameters,

$$\zeta_0 = 2R^{\frac{1}{2}}\Omega\left(1+\frac{\beta^2}{4k^2}\right)^{\frac{1}{2}}. \tag{23}$$

Thus at any point in the three-dimensional R, j, Ω space we can specify whether a critical level is partially, totally, or over-reflective to a gravity wave incident from below. Note, though, that the results are dependent on the wind structure above the critical level.

In addition, the quantitative reflectivity is dependent on the structure of the wind below the critical level, since there may be partial reflexions at lower levels. However, the qualitative results above are not influenced by these partial reflexions, as shown by the constancy of sign of $\overline{p_1 w_1}$ below the critical level.

The sign of $\overline{p_1 w_1}$ has been determined numerically over a range of $0 < R \leqslant \frac{1}{4}$, $0 \leqslant j \leqslant 1$ and $0 \leqslant \Omega \leqslant 5$. Whitaker functions were computed from equation (14), carrying the expansion to terms $< 10^{-8}$. As an inhomogeneous boundary condition, w_1 was set equal to unity at $\zeta = 0$.

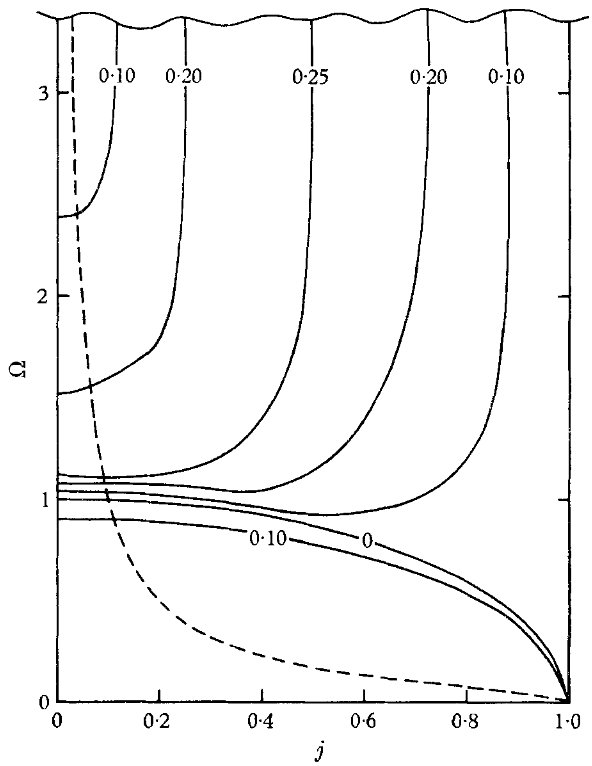

FIGURE 1. Richardson number contours of the surface of total reflectivity. Ω is the wave frequency non-dimensionalized in terms of the Brunt–Väisälä frequency and j is a measure of horizontal scale going from zero to one as wavelength goes from zero to infinity. At a given Ω and j, waves are over-reflected at lower Richardson number and partially reflected at higher Richardson number.

The results show two domains of over-reflexion, separated from a domain of partial reflexion by a surface of total reflectivity. R contours of this surface of total reflectivity are shown in figure 1. The surface dips to $R = 0$ along a curve, separating the over-reflective domains. This curve also separates two domains of behaviour of the transmitted wave: in the upper region the transmitted wave propagates vertically as an internal gravity wave in the upper half-space; in the lower region it is evanescent.

There is a discontinuity in the momentum flux at the critical level so that there is a mean transport of momentum between the level at which the wave is generated and the critical level. As these levels have different mean velocities, such a transport implies a net change in the total kinetic energy of mean flow.

This is the source of energy from which the wave can borrow to provide for over-reflexion.

In the 'evanescent' domain, the total reflectivity surface extends as high as $R = 0{\cdot}25$. Above this surface, the momentum flux is such that the wave surrenders energy to the mean flow. At lower R, it extracts energy.

In the 'propagating' domain, the total reflectivity surface approaches $R = 0{\cdot}115$ as an upper limit. At total reflectivity in this case there is a net output of wave energy. Incident input and reflected output balance, but there is now an additional transmitted energy flux. It is intuitively not surprising that greater shears are needed in order to extract this energy from the mean flow.

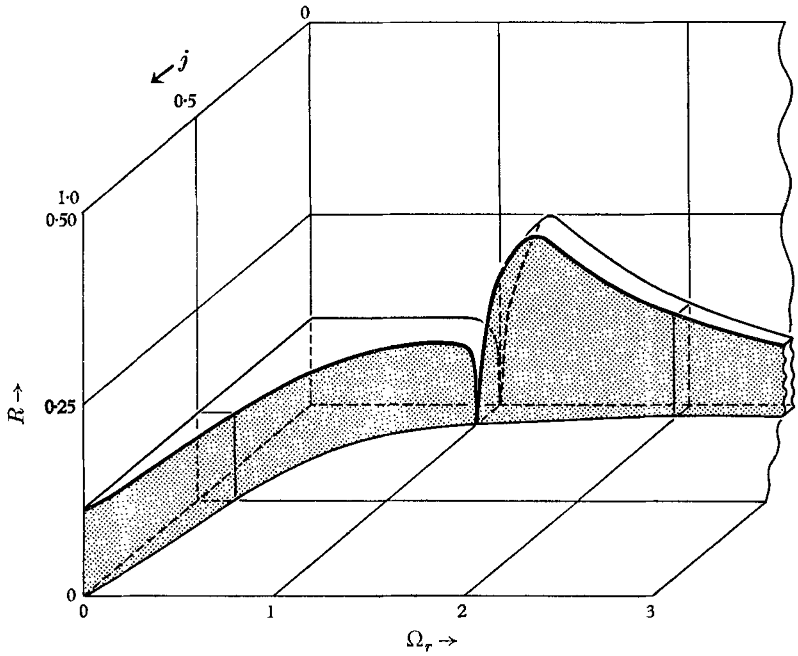

FIGURE 2. Domains of over-reflexion for waves incident from below on a critical level. The atmosphere is assumed to consist of two layers of uniform velocity on either side of a shear layer of finite thickness and velocity difference $\Delta u_0 = 0{\cdot}2g^{\frac{1}{2}}\beta^{\frac{1}{2}}$. Waves with values of R, j and Ω lying within the volume shown are over-reflected.

If the shear layer is restricted so that there is a finite velocity difference Δu_0 between the top and the bottom of the layer, there will be a further restriction on the over-reflective domains. This stems from the fact that at sufficiently large values of j and Ω there will be no critical level in the shear zone. Thus if

$$\Delta u_0 = 0{\cdot}2g^{\frac{1}{2}}\beta^{-\frac{1}{2}}, \qquad (24)$$

for example, the total reflectivity surface will be confined to the region below the dashed line in figure 1. A perspective view of the over-reflective domains in this example is shown in figure 2.

Suppose that this model is limited on the bottom by a totally reflecting lower boundary condition. This might take the form of a solid surface located at the bottom of the shear zone or separated from it by a region of constant mean velocity. It might also arise from an infinite lower half space of constant velocity in which the wave mode is evanescent.

Consider the possible singular neutral modes for such a model. A singular neutral mode is an eigensolution of real frequency having a critical level. From simple energetic considerations, it is obvious that singular neutral modes must lie in the surface of total reflectivity. Only then can $\overline{pw}$ be zero at the lower boundary.

The converse, that the surface of total reflectivity is a surface of singular neutral modes, is not true. Nor is it true that linearly unstable modes are confined to the over-reflexion domains.

3. Unstable normal modes

Numerical approach

Assume that a suitable lower boundary condition is applied to the model of a constant shear zone topped by an infinite half space of uniform flow. The model may then be treated as a homogeneous system and analysed for normal mode solutions. In particular, when Ω is complex, so that

$$\Omega = \Omega_r + i\Omega_i, \tag{25}$$

solutions with negative imaginary component of frequency are unstable. Case (1960) has shown that the normal mode solutions must be supplemented by non-exponential solutions, but Miles (1961) has pointed out that these are decaying for positive values of R. Thus the only unstable solutions are those with negative imaginary ω.

We have analysed three models to define numerically the unstable modes of oscillation. The shear zone and upper half space are taken as for the reflectivity study. The lower boundaries of these models are as follows.

Model I. The shear zone is terminated at the bottom by a solid horizontal surface at which $w_1 = 0$.

Model II. The lower boundary is again a solid horizontal surface, at which $w_1 = 0$, but there is an intermediate layer of uniform velocity equal to that at the bottom of the shear layer. This layer is taken to have a thickness $L = 0{\cdot}1/\beta$, one-tenth of a density scale height.

Model III. The shear layer is bounded beneath by an infinite half-space of uniform velocity equal to that of the bottom of the shear layer. In this case decaying or radiative boundary conditions are applied at $z = -\infty$. In both models II and III, interface boundary conditions must be applied.

In all three cases, the velocity difference across the shear layer is fixed at $\Delta u_0 = 0{\cdot}2 g^{\frac{1}{2}}\beta^{-\frac{1}{2}}$.

The boundary conditions lead to a set of algebraic equations, with a secular determinant

$$D(\Omega_r, \Omega_i, R, j, \Delta u_0 \beta^{\frac{1}{2}} g^{-\frac{1}{2}}), \tag{26}$$

which is equal to zero for a normal mode. For model II, there is an additional parameter, $L\beta$, which could be varied. We may pose the problem as follows. For what values of Ω_r, R, j, $\Delta u_0 \beta^{\frac{1}{2}}/g^{\frac{1}{2}}$ and $L\beta$ are there negative values of Ω_i such that $D = 0$? The unstable normal modes thus may form hypersurfaces in a four- or five-dimensional space.

In order to simplify the problem, we have fixed two of the parameters, $L\beta$ and $\Delta u_0 \beta^{\frac{1}{2}} g^{-\frac{1}{2}}$. The unstable normal modes then form ordinary surfaces in the three-dimensional Ω_r, R, j space.

The numerical procedure followed was a more or less brute force one. First particular values of Ω_r and R were chosen. The real and imaginary parts of the secular determinant were then evaluated over a range of j and Ω_i, normally 21 points in each of these two directions. Contours of the real and imaginary parts of the determinant were drawn automatically in j, Ω_i space and presented as photographic output from a Control Data dd 80 cathode-ray tube plotter attached to the Control Data 6600 computer. Any intersections of the zero contours of the real and imaginary components marked the locations for unstable normal modes. This procedure was repeated for new values of R and Ω_r. This was done for between 400 and 1300 combinations of R and j for each model, depending on the particular resolution needed, the ranges of j and Ω_i being varied to provide optimum resolution.

The results for each model are discussed below.

Model I

This model showed no normal mode instabilities for any Richardson number in the range of $0{\cdot}0025 \leqslant R \leqslant 0{\cdot}25$. This is of interest in the light of a conjecture by Kuo (1963). Synge's (1933) generalization of Rayleigh's inflexion point theorem (see also Howard 1961) is that a necessary condition for instability is that

$$(\rho_0 u_0')' - \frac{2k\beta g(\omega_r + ku_0)}{|\omega + ku_0|^2} \tag{27}$$

change sign somewhere within the fluid. If the fluid is statically neutrally stable so that $g\beta = 0$, then the necessary condition is simply that $(\rho_0 u_0')'$ change sign. Kuo suggests that in view of the stabilizing effect of stable stratification such an inflexion point might be a necessary condition for instability with positive R. This model is a crude representation of an atmosphere with smoothly varying u_0 and no inflexion point. On the other hand, models II and III are crude representations of atmospheres with inflexion points and are unstable.

Model II

The instability surface for this model is shown in perspective in figure 3. Except for one critical area, it is a more or less vertical sheet. (Values of Ω normally vary by a few per cent for fixed j over the height of the surface.) Near $\Omega_r = 1$, however, there is a cusp or flap that is folded over and returns to the $R = 0$ plane at $\Omega_r \simeq 1{\cdot}0$, $j \simeq 0{\cdot}069$. This cusp is shown in detail in a different projection in figure 4.

This surface has a stability boundary consisting of a singular neutral mode which lies in the surface of total reflectivity of figures 1 and 2. However, the

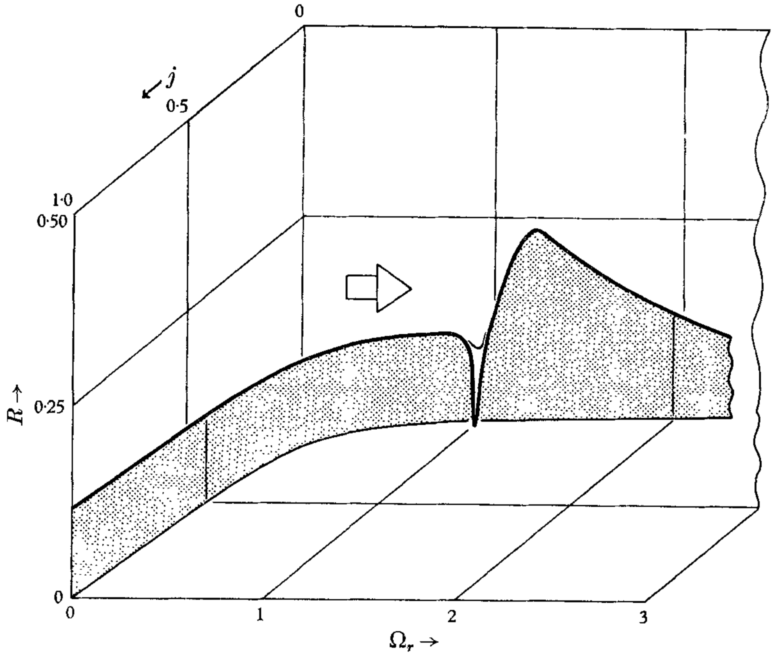

FIGURE 3. Instability surface for model II (earth, shear layer, uniform velocity layer). Normal modes with unstable imaginary frequency components are found throughout the shaded three-dimensional surface. Note the 'flap' in the surface folded over near $\Omega_r = 1$.

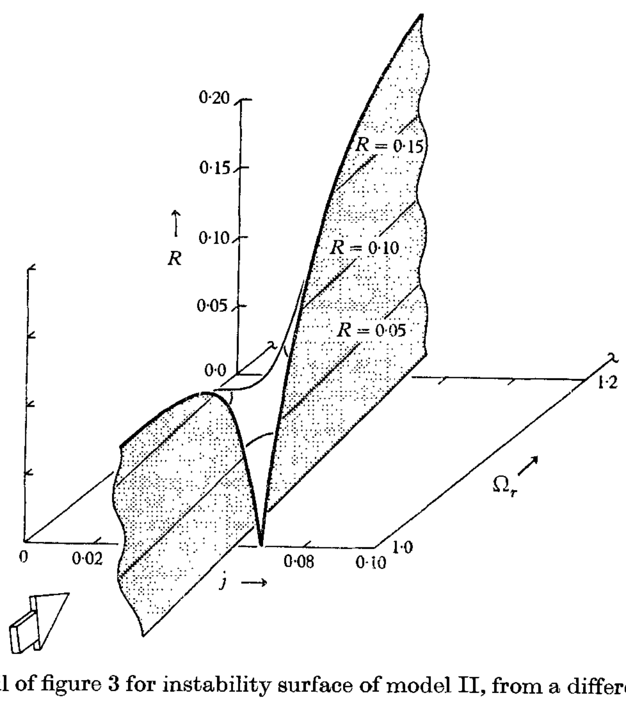

FIGURE 4. Detail of figure 3 for instability surface of model II, from a different perspective.

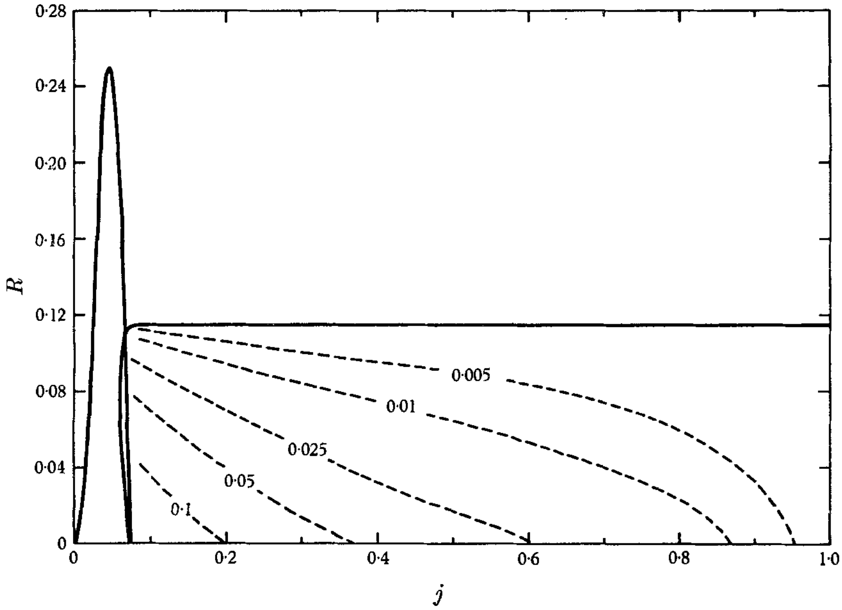

FIGURE 5. Projection of instability surface of model II on the (j, R)-plane. Contours of constant growth rate, Ω_i, are shown by dashed lines, and the projection of the singular neutral modes by the heavy solid line.

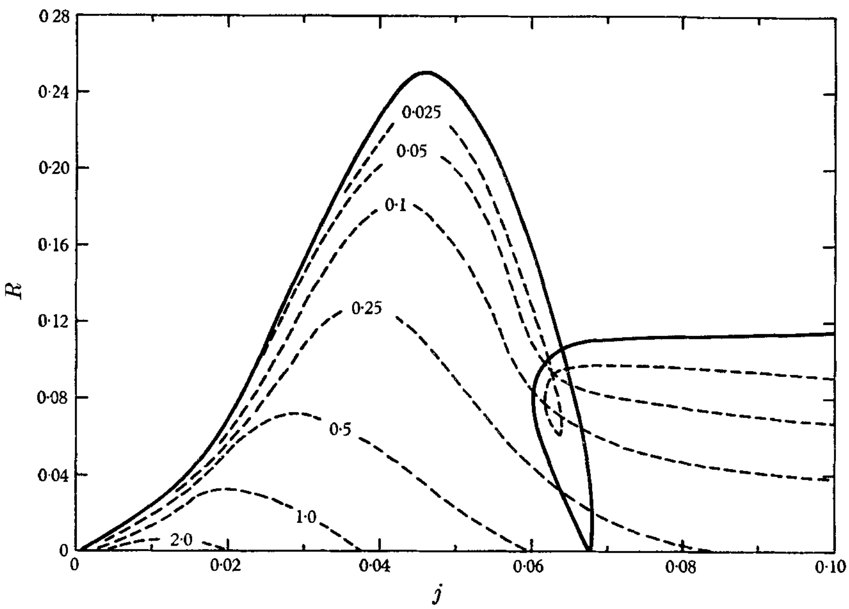

FIGURE 6. Expanded detail of figure 5.

instability surface is not divided into two sections as are the domains of over-reflexion. Also, the instability surface of figure 3 in some places intersects the surface of total reflectivity without the intersection being a singular neutral mode.

The projection of the instability surface on the (j, R)-plane is shown in figure 5 with detail shown in figure 6. Contours of constant Ω_i are also shown. If only the singular neutral mode projection had been drawn, it would be easy to conclude that there were two overlapping modes of instability. Only the continuity of the Ω_i contours reveals that there is one instability domain which is folded back upon itself.

Model III

The instability surface for this model is shown in figure 7 with detail shown in figure 8. This surface is indeed divided into two domains with bounding singular neutral modes. The projections of these instability surfaces on the j, R surface are shown on figures 9 and 10. At $j \leqslant 0{\cdot}05$, the surface behaves smoothly. The critical layer at which $\omega_r + ku_0 = 0$ is nearly in the centre of the shear zone, the small asymmetry stemming from the term involving j in equation (8). [In the Boussinesq approximation (Drazin 1958) this term is dropped. The hypothesis of exchange of stabilities then places the critical layer exactly in the centre.]

In order to understand the behaviour of the remainder of the instability surface, it is useful to consider the bounding singular neutral mode. At small values of j, hence at large wave-number k, the solutions are evanescent in both bounding half spaces. As j is increased, however, this becomes impossible. In the range $0{\cdot}05 < j < 0{\cdot}10$ there are two singular neutral modes, one evanescent in the upper half space, one in the lower half space.

As j is increased beyond $\sim 0{\cdot}10$, the wave-number becomes so small that it is no longer possible to find a solution evanescent in either domain. The frequency $k\Delta u_0$ drops below the upper cut-off frequency for propagating internal gravity waves. It appears that at positive R radiative losses to both plus and minus infinity are greater than can be sustained by energy extraction from the mean flow.

The singular neutral mode running from B to B' in figure 8 does not lie in the surface of total reflectivity, but is within the over-reflective domain. This mode has a radiative rather than evanescent boundary condition at $z = -\infty$, and the radiative energy loss must be accounted for.

Figures 9 and 10 show the projections of the two instability surfaces on the (j, R)-plane.

Discussion

There are several points that seem worthy of notice. The cusp, or flap, of figures 3 and 4 raises an interesting possibility. It might well be that given the right boundary conditions, a flap might exist whose projection on the (j, R)-plane would not be bounded by the projection of the singular neutral mode. This does occur for model II for the projection on the (Ω_r, R)-plane. Unless it can be shown that this is not possible, analysis in these two dimensions alone must be pursued with care.

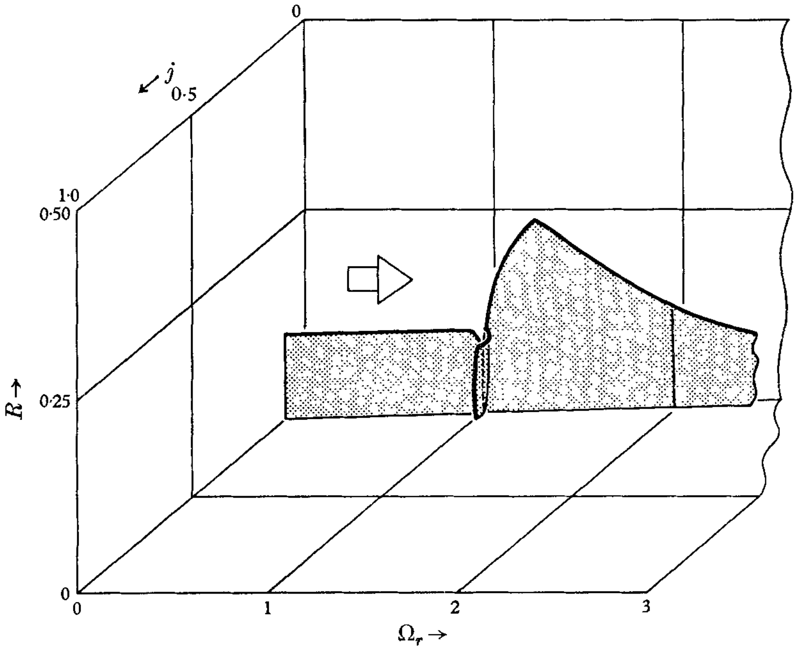

FIGURE 7. Instability surface for model III (shear layer bounded on both sides by uniform velocity layers). Unstable modes form the two shaded surfaces.

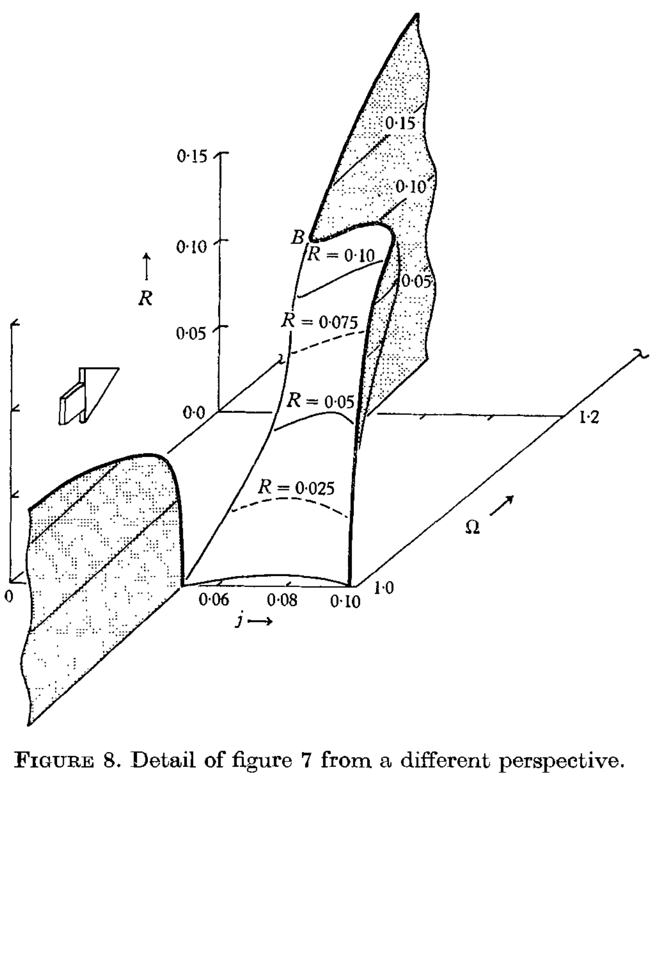

FIGURE 8. Detail of figure 7 from a different perspective.

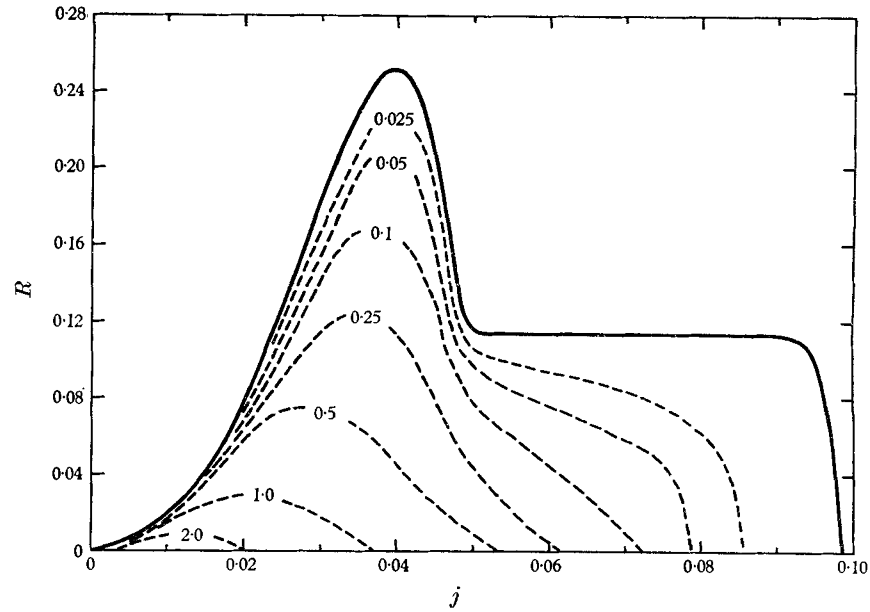

FIGURE 9. Projection of first instability surface of model III on the (j, R)-plane.

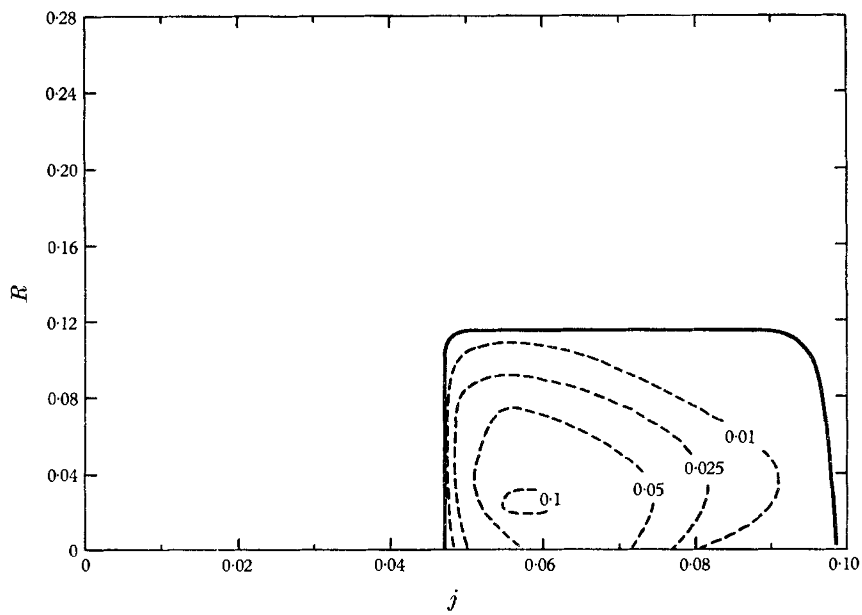

FIGURE 10. Projection of the second instability surface of model III on the (j, R)-plane.

The greatest instability of models II and III occurs at smaller values of j. If L is the layer thickness, then

$$kL = 0{\cdot}1R^{\frac{1}{2}}(j^{-2}-1)^{\frac{1}{2}}. \tag{28}$$

Empirically, the maximum growth rates at any given R appear to be at $kL \simeq 1$. This is only an approximate relation; the optimum value of kL decreases slowly as R is decreased. At larger wavelengths, a stabilizing influence is the efficiency with which the disturbance transports energy vertically away from the shear zone. Hence, growth rates are smaller. If the wavelength and boundary conditions are such that energy is efficiently transported in both directions, the models seem stable at any positive Richardson number.

Model III is a crude approximation to the model used by Drazin, who employed a wind profile of the form

$$u_0 = u_{00}\tanh z/d.$$

The singular neutral mode found by Drazin would correspond to a singular neutral mode lying in the right-hand surface of total reflectivity of figure 2. At $\Omega_r > 1{\cdot}2$, our results give a good approximation to this. However, when the disturbance non-dimensional real frequency is close to unity when referred to a frame of reference moving with the fluid at *either* $z \to \infty$ or $z \to -\infty$, the vertical scale of the disturbance in this region is large, and the Boussinesq approximation used by Drazin is no longer valid. (This corresponds to the condition that $kd \to 1$.) It is in this non-Boussinesq régime that our results differ from Drazin's.

Finally, these results give further credence to the belief that there should be some form of inflexion criterion as a necessary condition for instability in a stably stratified fluid.

4. Application to a compressible atmosphere

These results have been derived on the assumption of an incompressible density-stratified fluid. One may question whether these results may be applied to a compressible fluid. Yih (1965) has considered this question for an isothermal atmosphere. The following argument is based on his approach.

For an isothermal compressible atmosphere with no mean wind, the counterpart to (5) is

$$\frac{\partial^2 w}{\partial z^2} - \beta w + k^2\left[\frac{N'^2}{\omega^2} - 1 + \frac{\omega^2}{c_s^2 k^2}\right]w = 0, \tag{29}$$

where c_s is the speed of sound and

$$N'^2 = \left(g\beta - \frac{g^2}{c_s^2}\right) \tag{30}$$

is a modified Brunt–Väisälä frequency. Under these constraints, (5) is

$$\frac{\partial^2 w}{\partial z^2} - \beta w + k^2\left[\frac{N^2}{\omega^2} - 1\right]w = 0 \tag{31}$$

for a compressible fluid.

There are two important differences to note. A modified value of N^2 must be substituted in the compressible case, and the term $\omega^2/c_s^2 k^2$ must be considered. This term is the square of the ratio of wave horizontal phase speed to sound speed. It is normally quite small as long as $u_0 \ll c_s$. Except when $\omega^2 \rightarrow N'^2$, this term will be negligible. Even under this condition, the behaviour of the two equations is similar; the vertical wavelength increases and the wave becomes evanescent as ω is increased. The transition from a transmitting to a reflecting top is obtained at slightly different frequencies.

On this basis we anticipate that when an appropriate value of the Brunt–Väisälä frequency is applied, our results will be qualitatively correct for an isothermal compressible atmosphere. The quantitative differences will be most pronounced close to the transition from a transmitting to a reflecting model top.

William Taffe laid much of the groundwork for the reflectivity analysis while on a summer visit to NCAR, while Larry Williams and Jeanne Adams gave a great deal of help to the author in performing the numerical computations.

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